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The exact Green function of a relativistic Coulomb system via the transformation method

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Abstract. The exact Green function of a relativistic Coulomb system is given by the transformation method. The earlier treatments are based on the multiple-valued transformation of Kustaanheimo and Stiefel as well as the perturbation expansions. The method presented in this paper relates the relativistic Coulomb path integral to the simple ones and may apply to a large class of problems.

1. Introduction

In this paper, we apply the transformation method [1, 2] to the path integral representation of Kleinert for the relativistic potential problems [3, 4] and calculate the Green function of the relativistic Coulomb system. The earlier treatment of the same system in [6] is solved by the path integral with the space–time and Kustaanheimo–Stiefel (KS) transformations and in [7] it is calculated via the perturbation technique. The method presented here combines the space–time and coordinate transformation of the general case to form a new path integral representation. It can relate the unknown relativistic path integral to the known one and solve it by changing the variables of known solutions of the path integral. The procedure just involves a search of the transformation functions and is applicable for arbitrary one-dimensional and spherical symmetry systems.

2. Duru–Kleinert equivalence of the relativistic path integral

The fixed-energy Green function of a relativistic particle moving in external potentials from $\boldsymbol{x}_{\mathrm{a}}$ to $\boldsymbol{x}_b$ of the spatial part of the $(D+1)$ vector $x^\mu = (\boldsymbol{x}(\lambda), \tau(\lambda))$ can be expressed as the local matrix element

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \langle \boldsymbol{x}_{\mathrm{b}} | \hat{R} | \boldsymbol{x}_a \rangle \tag{1}$$

of the resolvent operator

$$\hat{R} = \frac{\mathrm{i}\hbar}{2mc} \frac{1}{(\hat{H}_E - \varepsilon)} \tag{2}$$

where i is the imaginary number by convention, $\hbar/mc$ is the well known Compton wavelength of a particle of mass m, the energy dependent Hamiltonian operator is $\hat{H}_E = \hat{\boldsymbol{p}}^2/2m + \hat{V}_E$ in which the potential operator $\hat{V}_E$ is defined as $(V^2 - 2EV)/2mc^2$ with $V(\boldsymbol{x})$ representing the

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scalar potential of the systems and E is the system energy. Finally, ε is the pseudoenergy, defined as $(m^2c^4 - E^2)/2mc^2$. Due to the additional freedom of the parameter describing the relativistic orbits, the resolvent operator $\hat{R}$ has the other representations

$$\hat{R} = \frac{\mathrm{i}\hbar}{2mc}\frac{1}{\hat{\rho}_\mathrm{l}(\hat{H}_E - \varepsilon)}\hat{\rho}_\mathrm{l} \tag{3}$$

$$\hat{R} = \frac{\mathrm{i}\hbar}{2mc}\hat{\rho}_\mathrm{r}\frac{1}{(\hat{H}_E - \varepsilon)\hat{\rho}_\mathrm{r}} \tag{4}$$

or in more general form

$$\hat{R} = \frac{\mathrm{i}\hbar}{2mc}\hat{\rho}_\mathrm{r}\frac{1}{\hat{\rho}_\mathrm{l}(\hat{H}_E - \varepsilon)\hat{\rho}_\mathrm{r}}\hat{\rho}_\mathrm{l} \tag{5}$$

in which the operators $\hat{\rho}_\mathrm{l}$, $\hat{\rho}_\mathrm{r}$ are arbitrary operators and may depend on $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{x}}$. With these considerations, the relativistic Green function in equation (1) can be expressed as the following representation:

$$G(\boldsymbol{x}_\mathrm{b}, \boldsymbol{x}_\mathrm{a}; E) = \langle \boldsymbol{x}_\mathrm{b}|\hat{R}|\boldsymbol{x}_\mathrm{a}\rangle = \frac{\mathrm{i}\hbar}{2mc}\int_{\lambda_\mathrm{a}}^{\infty} \mathrm{d}\lambda_\mathrm{b}\langle \boldsymbol{x}_\mathrm{b}|\hat{U}_\varepsilon(\lambda_b - \lambda_\mathrm{a})|\boldsymbol{x}_\mathrm{a}\rangle \tag{6}$$

where the pseudo-evolution operator $\hat{U}_\varepsilon$ is given by

$$\hat{U}_\varepsilon(\lambda_\mathrm{b} - \lambda_a) = \hat{\rho}_\mathrm{r}\mathrm{e}^{-(\lambda_b - \lambda_\mathrm{a})\hat{\rho}_\mathrm{l}(\hat{H}_E - \varepsilon)\hat{\rho}_r}\hat{\rho}_\mathrm{l}. \tag{7}$$

With the help of the time-slice technique, the relativistic Green function can therefore be calculated from the path integral [6]:

$$G(\boldsymbol{x}_\mathrm{b}, \boldsymbol{x}_\mathrm{a}; E) = \frac{\hbar\mathrm{i}}{2mc}\int_0^{\infty} \mathrm{d}S \int \mathcal{D}\rho(\lambda)\Phi[\rho(\lambda)] \int \mathcal{D}^D x(\lambda)\exp\{-\mathcal{A}_E[\boldsymbol{x}, \dot{\boldsymbol{x}}]/\hbar\}\rho(0) \tag{8}$$

where $S = (\lambda_\mathrm{b} - \lambda_\mathrm{a})$ and the action

$$\mathcal{A}_E[\boldsymbol{x}, \dot{\boldsymbol{x}}] = \int_{\lambda_a}^{\lambda_\mathrm{b}} \mathrm{d}\lambda \left[\frac{m}{2\rho(\lambda)}\dot{\boldsymbol{x}}^2(\lambda) - \rho(\lambda)\frac{(E - V(\boldsymbol{x}))^2}{2mc^2} + \rho(\lambda)\frac{mc^2}{2}\right]. \tag{9}$$

Here $\rho(\lambda) = \rho_\mathrm{l}(\lambda)\rho_\mathrm{r}(\lambda)$, $\rho(0)$ is the terminal point of the function $\rho(\lambda)$ and $\Phi[\rho(\lambda)]$ is some convenient gauge-fixing functional. The only condition on $\Phi[\rho(\lambda)]$ is that

$$\int \mathcal{D}\rho(\lambda)\Phi[\rho(\lambda)] = 1. \tag{10}$$

Equation (8) has an equivalent representation providing the new path integral solutions via well known ones if the potential problems are in two-dimensional Minkowski space or possess rotational invariance in any dimension. This is given by observing the function $\rho(x)$, which brings the kinetic term to an inconvenient form containing a space-dependent mass $m/\rho(x)$. This space dependence is removed by a coordinate transformation $x = h(q)$. Since the coordinate differentials are related by $\mathrm{d}x = h'(q)\,\mathrm{d}q$, we require the function $h(q)$ to satisfy $h'^2(q) = \rho(h(q))$. Then the action in equation (9) turns into

$$\mathcal{A}_E^q[q, \dot{q}] = \int_{\lambda_\mathrm{a}}^{\lambda_b} \mathrm{d}\lambda \left[\frac{m}{2}\dot{q}^2(\lambda) - \rho(q(\lambda))\frac{(E - V(q(\lambda)))^2}{2mc^2} + \rho(q(\lambda))\frac{mc^2}{2}\right] \tag{11}$$

with the obvious notation $\rho(q) = \rho(h(q))$, $V(q) = V(h(q))$. When we combine the measure transformation of $x = h(q)$, the important fact is that the fixed-energy Green function in equation (8) can be related to the new kind of Green function by extending the action of equation (11) to an effective potential $V_\mathrm{eff}(q(s))$. This provides us with a powerful method to evaluate the relativistic path integral via the known one.

For relativistic systems with spherical symmetries, it is given by [3,4]

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{1}{(r_{\rm b} r_a)^{(D-1)/2}} \sum_{l=0}^{\infty} G_l^{\rm N}(r_{\rm b}, r_{\rm a}; E) \sum_{\hat{m}} Y_{l\hat{m}}(\hat{\boldsymbol{x}}_b) Y^*_{l\hat{m}}(\hat{\boldsymbol{x}}_a) \tag{12}$$

where the functions $Y_{l\hat{m}}(\hat{x})$ are the D-dimensional hyperspherical harmonics and $G_l^{\rm N}(r_{\rm b}, r_{\rm a}; E)$ is the *new* radical transformed Green function. It reads

$$G_l^{\rm N}(r_{\rm b}, r_{\rm a}; E) = \frac{\hbar {\rm i}}{2mc} \rho_{\rm b}^{1/4} \rho_a^{1/4} G(q_{b,} q_{\rm a}; \mathcal{E}) \tag{13}$$

with the Green function $G(q_{b,} q_{\rm a}; \mathcal{E})$ of the fixed pseudoenergy $\mathcal{E}$

$$G(q_{\rm b}, q_{\rm a}; \mathcal{E}) \equiv \int_0^{\infty} {\rm d}S \int \mathcal{D}q(\lambda) {\rm e}^{-\mathcal{A}^{\rm N}[q,\dot{q}]/\hbar} \tag{14}$$

in which

$$\mathcal{A}^{\rm N}[q, \dot{q}] = \int_{\lambda_{\rm a}}^{\lambda_b} {\rm d}\lambda \left[\frac{m}{2} \dot{q}^2(\lambda) + \rho(q(\lambda)) \left(\frac{\hbar^2}{2m} \frac{(l + D/2 - 1)^2 - 1/4}{r^2(q(\lambda))} \right.\right.$$
$$\left.\left. - \frac{[E - V(r(q))]^2}{2mc^2} + \frac{mc^2}{2} \right) + V_{\rm eff}(q(\lambda)) \right]. \tag{15}$$

The effective potential $V_{\rm eff}$ is given by [1–5]

$$V_{\rm eff}(q(\lambda)) = -\frac{\hbar^2}{m} \left[\frac{1}{4} \frac{h'''(q)}{h'(q)} - \frac{3}{8} \left(\frac{h''(q)}{h'(q)} \right)^2 \right] \tag{16}$$

with $h'(q)$ representing the derivative ${\rm d}h(q)/{\rm d}q$ and the transformation function $h(q)$ defined as $r = h(q)$, which is related to the local gauge transformation function $\rho(r)$ by the following equality:

$$h'^2(q) = \rho(r). \tag{17}$$

The detailed time-slice analysis can be found in chapter 14 of [1] and [5].

3. Green function of the relativistic Coulomb system by the transformation method

Let us now apply the equivalent relation of equation (13) to calculate the exact Green function of the relativistic Coulomb system. For such a system as under consideration, the potential $V(r_{\rm C}) = -{\rm e}^2/r_{\rm C}$ and the relativistic radial path integral in equation (8) reads [3,4,8]

$$G_{l_{\rm C}}(r_{\rm Cb}, r_{\rm Ca}; E_{\rm C}) = \frac{\hbar {\rm i}}{2m_{\rm C} c} \int_0^{\infty} {\rm d}S \int \mathcal{D}\rho(\lambda) \Phi[\rho(\lambda)] \int \mathcal{D}r_{\rm C}(\lambda) \exp\left\{ -\frac{1}{\hbar} \mathcal{A}_l[r_{\rm C}, r'_{\rm C}] \right\} \tag{18}$$

with the action

$$\mathcal{A}_l[r_{\rm C}, r'_{\rm C}] = \int_{\lambda_a}^{\lambda_{\rm b}} {\rm d}\lambda \left[\frac{m_{\rm C}}{2\rho(\lambda)} r'^2_{\rm C}(\lambda) + \frac{\rho(\lambda)\hbar^2}{2m_{\rm C}} \frac{(l_{\rm C} + D_{\rm C}/2 - 1)^2 - 1/4}{r_{\rm C}^2} \right.$$
$$\left. \times - \rho(\lambda) \frac{(E_{\rm C} + {\rm e}^2/r_{\rm C})^2}{2m_{\rm C}c^2} + \rho(\lambda) \frac{m_{\rm C}c^2}{2} \right]. \tag{19}$$

The Roman subscript C specifies the Coulomb system. Let us apply the transformation formula (13) to this relativistic system by taking the following transformation variables:

$$\begin{aligned} r_{\rm C} &= h(x) = {\rm e}^x \\ h'^2(x) &= {\rm e}^{2x} = \rho(r_{\rm C}) = r_{\rm C}^2 \end{aligned} \tag{20}$$

which maps the interval $r \in (0, \infty)$ into $x \in (-\infty, \infty)$ and leads to the effective potential

$$V_{\rm eff}(x(s)) = \frac{\hbar^2}{8m_{\rm C}}; \tag{21}$$

then the transformed Green function (13) turns into the non-relativistic Morse potential system:

$$\begin{aligned} G^{\rm N}_{l_{\rm C}}(r_{\rm Cb}, r_{\rm Ca}; E_{\rm C}) &= \frac{\hbar {\rm i}}{2m_{\rm C}c}\rho_{\rm b}^{1/4}\rho_{\rm a}^{1/4} G(x_{b,}x_a; \mathcal{E}_{\rm M}) \\ &= \frac{\hbar {\rm i}}{2m_{\rm C}c}{\rm e}^{x_b/2}{\rm e}^{x_a/2}\int_0^\infty {\rm d}S \int \mathcal{D}x(s){\rm e}^{-\mathcal{A}^{\rm N}[x,\dot{x}]/\hbar} \end{aligned} \tag{22}$$

with the action

$$\mathcal{A}^{\rm N}[x, \dot{x}] = \int_0^S {\rm d}s \left[\frac{m_{\rm C}}{2}\dot{x}^2(s) + \frac{v^2\hbar^2}{2m_{\rm C}}({\rm e}^{2x} - 2\beta{\rm e}^x) - \mathcal{E}_{\rm M}\right]. \tag{23}$$

The parameters associated with the relativistic Coulomb system are given as

$$\begin{aligned} v &= \frac{1}{\hbar c}\sqrt{m_{\rm C}^2c^4 - E_{\rm C}^2} \\ \beta &= \frac{E_{\rm C}{\rm e}^2}{m_{\rm C}^2c^4 - E_{\rm C}^2} \\ \mathcal{E}_{\rm M} &= -\frac{\hbar^2}{2m_{\rm C}}[(l_{\rm C} + D_{\rm C}/2 - 1)^2 - \alpha^2] \end{aligned} \tag{24}$$

where the notation $\alpha = {\rm e}^2/\hbar c$ is the fine-structure constant. To proceed further, let us make a trivial transformation by taking

$$\begin{aligned} x &= 2x_{\rm O} \\ m_{\rm C} &= m_{\rm O}/4. \end{aligned} \tag{25}$$

This maps equation (22) into

$$G^{\rm N}_{l_{\rm C}}(r_{\rm Cb}, r_{\rm Ca}; E_{\rm C}) = \frac{\hbar {\rm i}}{2m_{\rm C}c}{\rm e}^{x_{\rm Ob}}{\rm e}^{x_{\rm Oa}}\frac{1}{2}\int_0^\infty {\rm d}S \int \mathcal{D}x_{\rm O}(s){\rm e}^{-\mathcal{A}^{\rm N}[x_{\rm O},\dot{x}_{\rm O}]/\hbar} \tag{26}$$

with the transformed new action

$$\mathcal{A}^{\rm N}[x_{\rm O}, \dot{x}_{\rm O}] = \int_0^S {\rm d}s \left[\frac{m_{\rm O}}{2}\dot{x}_{\rm O}^2(s) + \frac{2v^2\hbar^2}{m_{\rm O}}({\rm e}^{4x_{\rm O}} - 2\beta{\rm e}^{2x_{\rm O}}) - \mathcal{E}_{\rm M}\right]. \tag{27}$$

The factor $\frac{1}{2}$ in equation (26) accounts for the fact that the normalized states are related by $|x\rangle = |x_{\rm O}\rangle/2$. At this point, we can apply the transformation method again by taking the following transformation functions:

$$\begin{aligned} x_{\rm O} &= \ln z = h(z) \\ h'^2(z) &= 1/z^2 = \rho(x_{\rm O}) = {\rm e}^{-2x_{\rm O}} \end{aligned} \tag{28}$$

which maps the interval $x_{\rm O} \in (-\infty, \infty)$ into $z \in (0, \infty)$ and leads the effective potential $V_{\rm eff}(z)$ to

$$V_{\rm eff}(z) = -\frac{\hbar^2}{8m_{\rm O}z^2}. \tag{29}$$

The Green function in equation (26) becomes

$$G^{\rm N}_{l_{\rm C}}(r_{\rm Cb}, r_{\rm Ca}; E_{\rm C}) = \frac{\hbar {\rm i}}{2m_{\rm C}c}\frac{1}{2}z_{\rm b}z_{\rm a}\left\{\frac{1}{\sqrt{z_{\rm b}z_{\rm a}}}\int_0^\infty {\rm d}S' \int_0^\infty \mathcal{D}z(\tau){\rm e}^{-\mathcal{A}^{\rm N}[z,\dot{z}]/\hbar}\right\} \tag{30}$$

with the action of the radial simple harmonic oscillator

$$\mathcal{A}^{\mathrm{N}}[z,\dot{z}]=\int_0^{S'}\mathrm{d}\tau\left[\frac{m_{\mathrm{O}}}{2}z^2(\tau)+\frac{\hbar^2}{2m_{\mathrm{O}}}\frac{(l_{\mathrm{O}}+D_{\mathrm{O}}/2-1)^2-1/4}{z^2}+\frac{m_{\mathrm{O}}\omega^2z^2}{2}-\mathcal{E}_{\mathrm{O}}\right]. \tag{31}$$

The parameter relations between the equations (26) and (31) are given as

$$\begin{aligned}&\hbar^2(l_{\mathrm{O}}+D_{\mathrm{O}}/2-1)^2=-2m_{\mathrm{O}}\mathcal{E}_{\mathrm{M}}\\&m_{\mathrm{O}}\omega^2/2=2v^2\hbar^2/m_{\mathrm{O}}\\&\mathcal{E}_{\mathrm{O}}=4\beta v^2\hbar^2/m_{\mathrm{O}}.\end{aligned} \tag{32}$$

By inserting the relations in equation (24) into these equalities, we obtain the parameter relations between the relativistic Coulomb and radial harmonic oscillator

$$\begin{aligned}&\mu_{\mathrm{O}}=2\sqrt{\mu_{\mathrm{C}}^2-\alpha^2}\\&\omega=\sqrt{m_{\mathrm{C}}^2c^4-E_{\mathrm{C}}^2}/2m_{\mathrm{C}}c\\&\mathcal{E}_{\mathrm{O}}=E_{\mathrm{C}}\mathrm{e}^2/m_{\mathrm{C}}c^2\end{aligned} \tag{33}$$

where for simplicity the quantities $(l_{\mathrm{O}}+D_{\mathrm{O}}/2-1)$ and $(l_{\mathrm{C}}+D_{\mathrm{C}}/2-1)$ have been defined as μ_{O} and μ_{C}, respectively. With the well known Green function of the radial harmonic oscillator

$$\begin{aligned}G_{l_{\mathrm{o}}}(z_{\mathrm{b}},z_{\mathrm{a}};\mathcal{E}_{\mathrm{O}})=&-\mathrm{i}\frac{1}{\omega}\frac{\Gamma((1+\mu_{\mathrm{O}})/2-\mathcal{E}_{\mathrm{O}}/2\hbar\omega)}{\Gamma(1+\mu_{\mathrm{O}})\sqrt{z_{\mathrm{b}}z_{\mathrm{a}}}}\\&\times W_{\mathcal{E}_{\mathrm{O}}/2\hbar\omega,\mu_{\mathrm{O}}/2}((m_{\mathrm{O}}\omega/\hbar)z_{\mathrm{b}}^2)M_{\mathcal{E}_{\mathrm{O}}/2\hbar\omega,\mu_{\mathrm{O}}/2}((m_{\mathrm{O}}\omega/\hbar)z_{\mathrm{a}}^2)\end{aligned} \tag{34}$$

we obtain the exact Green function of the relativistic Coulomb system in any dimensions:

$$\begin{aligned}G_{l_{\mathrm{c}}}(r_{\mathrm{Cb}},r_{\mathrm{Ca}}E_{\mathrm{C}})=&\frac{m_{\mathrm{C}}c}{\sqrt{m_{\mathrm{C}}^2c^4-E_{\mathrm{C}}^2}}\\&\times\frac{\Gamma\left(1/2+\sqrt{(l_{\mathrm{C}}+D_{\mathrm{C}}/2-1)^2-\alpha^2}-E_{\mathrm{C}}\alpha/\sqrt{m_{\mathrm{C}}^2c^4-E_{\mathrm{C}}^2}\right)}{\Gamma\left(1+2\sqrt{(l_{\mathrm{C}}+D_{\mathrm{C}}/2-1)^2-\alpha^2}\right)}\\&\times W_{E_{\mathrm{C}}\alpha/\sqrt{m_{\mathrm{C}}^2c^4-E_{\mathrm{C}}^2},\sqrt{(l_{\mathrm{C}}+D_{\mathrm{C}}/2-1)^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{m_{\mathrm{C}}^2c^4-E_{\mathrm{C}}^2}r_{\mathrm{Cb}}\right)\\&\times M_{E_{\mathrm{C}}\alpha/\sqrt{m_{\mathrm{C}}^2c^4-E_{\mathrm{C}}^2},\sqrt{(l_{\mathrm{C}}+D_{\mathrm{C}}/2-1)^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{m_{\mathrm{C}}^2c^4-E_{\mathrm{C}}^2}r_{\mathrm{Ca}}\right).\end{aligned} \tag{35}$$

The result in the three-dimensional case is first given in [6] by performing the path integral with KS transformation and later in [7] in any dimensions by summing the perturbation expansions.

4. Concluding remarks

In this paper, the transformation method is applied to the relativistic path integral. As an interesting application, the Green function of the relativistic Coulomb system is solved by the method. Different from the path integral approach [6] and the perturbation approach [7], the procedures presented in this paper just need to find the appropriate transformation functions. Furthermore, all one-dimensional systems and any higher-dimensional system with rotationally invariant systems are applicable. It is our hope that the method presented here may offer us a new way to solve the relativistic potential problems.

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